SOME GENERALIZED CENTRAL SETS THEOREM NEAR ZERO ALONG PHULARA'S WAY

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ABSTRACT. The central sets theorem near zero originally proved by M. Akbari Tootkaboni and E. Bayatmanesh. In this paper, we provide a generalization of the near zero version of Central Sets Theorem along Dev Phulara's way with some applications along two more generalized versions of the Central Sets theorem.

1. INTRODUCTION

The history of Central sets is a long story to study. It uses special techniques from topological dynamics, ergodic theory, the algebra of Stone-Čech compactification, combinatorics etc to study Central sets and other Ramsey theoretic results. It was H. Furstenberg who defined the Central sets using the notions from topological dynamics and proved the Central Sets theorem dynamically in [**?**].

Theorem 1. *(The original Central Sets theorem) Let A be a central subset of $\mathbb{N}$, let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, k\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) For each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$, and
(2) For each $i \in \{1, 2, \ldots, k\}$, and $F \in \mathcal{P}_f(\mathbb{N})$, we have

$$\sum_{n \in F} \left(a_n + \sum_{t \in H_n} y_{i,t} \right) \in A.$$

Proof. See [**?**]. □

Though Central sets was defined dynamically, there is an algebraic counterpart of this definition, was established by V. Bergelson and N. Hindman in [**?**]. Thereafter many versions of Central Sets theorem came, among which the following one is the one for any arbitrary semigroup.

Theorem 2. *Let $(S, \cdot)$ be a semigroup , let A be a central set in S, and for each $\ell \in \mathbb{N}$, let $\langle y_{\ell,n} \rangle_{n=1}^{\infty}$ be a sequence in S. There exist sequences $\langle m(n) \rangle_{n=1}^{\infty}$, $\langle \langle a_n(i) \rangle_{i=1}^{m(n)+1} \rangle_{n=1}^{\infty}$, and $\langle \langle H_n(i) \rangle_{i=1}^{m(n)} \rangle_{n=1}^{\infty}$ such that*
(1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$;
(2) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \cdots, m(n)+1\}$, $a_n(i) \in S$;
(3) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \cdots, m(n)\}$, $H_n(i) \in \mathcal{P}_f(\mathbb{N})$;
(4) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \cdots, m(n)-1\}$, max $H_n(i) <$ min $H_{n+1}(i+1)$;

(5) for each $n \in \mathbb{N}$, max $H_n\left(m(n)\right)<$ min $H_n\left(1\right)$; and
(6) for each $F \in \mathcal{P}_f\left(\mathbb{N}\right)$ and each $f \in \Phi = \{f \in^{\mathbb{N}} \mathbb{N} : f(n) \leq n \, for \, all \, n \in \mathbb{N}\}$,

$$\prod_{n\in F}\left(\prod_{i=1}^{m(n)}\left(a_n\left(i\right)\cdot\prod_{t\in H_n(i)}y_{f(n),t}\right)\cdot a_n\left(m(n)+1\right)\right)\in A.$$

Proof. [?]Theorem 14.14.9. □

Now we need to introduce some development towards algebra. Let $(S,+)$ be a discrete semigroup. The collection of all ultrafilters on S is called the Stone-Čech compactification of S and denoted byβS . For $A \subseteq S$, define $\bar{A} = \{p \in \beta S : A \in p\}$, then $\{\bar{A} : A \subseteq S\}$ is a basis for the open sets (also for the closed sets) of βS. There is a unique extension of the operation to βS making $(\beta S,+)$ a right topological semigroup (i.e. for each $p \in \beta S$, the right translation ρ_p is continuous where $\rho_p = q+p$) and also for each $x \in S$, left translation λ_x is continuous where $\lambda_x(q) = x+q$. The principal ultrafilters being identified with the points of S and S is a dense subset of βS . Given $p,q \in \beta S$ and $A \subseteqq S$, we have $A \in p+q$ if and only if $\{x \in S : -x+A \in q\} \in p$, where $-x+A = \{y \in S : x+y \in A\}$.

For $A \subseteq S$, and $p \in \beta S$, we define$A^*(p) = \{s \in A : -s+A \in p\}$.

We now discuss a different notion. $\mathcal{F}$will denote a filter of $(S,\cdot)$. For every filter $\mathcal{F}$of S, define $\bar{\mathcal{F}} \subseteq \beta S$, by

$$\bar{\mathcal{F}} = \bigcap_{V\in\mathcal{F}}\bar{V}.$$

It is a routine check that $\bar{\mathcal{F}}$ is a closed subset of βS consisting of ultrafilters which contain $\mathcal{F}$. If $\mathcal{F}$ is an idempotent filter, i.e., $\mathcal{F} \subset \mathcal{F}\cdot\mathcal{F}$, then $\bar{\mathcal{F}}$becomes a closed subsemigroup of βS, but the converse is not true. We will consider only those filters $\mathcal{F}$, for which $\bar{\mathcal{F}}$ is a closed subsemigroup of βS. For more results on study along filters see [?].

Centra sets theorem was studied near idempotents of 0 by M. Akbari Tootkaboni and E. Bayatmanesh in [?]. Dev Phulara generalized the theorem and proved that the conclution of the theorem is true for not only a single set but for a sequence of sets in [?]. Here we generalize the near zero version along Phulara's way, as well as of the more stronger version by De,Hindman and Strauss done in [?]. We also add the generalized version of Central sets theorem along filter done by Goswami and Poddar [?] but in the way of Phulara. We will need some lemma and introduction of notion of near zero first.

Lemma 3. *Let $(S,+)$ be a semigroup, let p is idempotent of S and let $p+p=p \in \beta S$. For each $s \in A^*(p), -s+A^*(p) \in p$*

Proof. [?], Lemma 4.14 □

The set 0^+ of all non principal ultrafilters on $S = ((0,+\infty),+)$ that is convergent to 0 is a semigroup under the restriction of the usual $'+'$ on βS, the Stone-Čech compactification of the discrete semigroup $S = ((0,+\infty),+)$ .We have been considering semigroups which are dense in $S = ((0,+\infty),+)$ with natural topology. When passing to the Stone-Čech compactification of such a semigroup S, we deal with S_d, which is the set S with the discrete topology.

Definition 4. Let S be a dense subset of $((0,+\infty),+)$. Then
$0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0), (0,\epsilon)\cap S \in p\}$.

We have to recall the notions of thick near zero, syndetic near zero and piecewise syndetic near zero.

Definition 5. Let S be a dense subsemigroup of $((0,+\infty),+)$. and let $A \subseteq S$.

(a) A is thick near zero if and only if $\exists \epsilon > 0)(\forall F \in P_f((0,\epsilon) \cap S)(\forall \delta > 0)(\exists y \in (0,\epsilon) \cap S)(F + y \subseteq A)$

(b) A is syndetic near zero if and only if for any $\epsilon > 0$ there exist $F \in P_f((0,\epsilon) \cap S$ and $\delta > 0$ such that $(0,\delta) \cap S \subseteq \cup_{t\in F} -t + A$.

(c) A is piecewise syndetic near zero if and only if for all $\delta > 0$ there exists $F \in P_f((0,\delta) \cap S$ such that $\cup_{t\in F} -t + A$ be thick near zero.

(d) A is a central set near zero if and only if there exists an idempotent p in the smallest ideal of $0^+(S)$ with $A \in P$.

In this paper, the minimal ideal in $0^+(S)$ is denoted by K.

Theorem 6. *Let $A \subseteq S$. Then $K \cap \bar{A} \neq \emptyset$ if and only if A is piecewise syndetic near* 0.

Proof. [?],Theorem 2.4 □

Definition 7. Let S be a dense subsemigroup of $((0,+\infty),+)$.

(1) We say that $f : \mathbb{N} \to \mathbb{S}$ is near zero if $\inf(f(\mathbb{N})) = 0$. The collection of all functions that is near zero is denoted by τ_0.
(2) Of course, we can say that $A \subseteq S$ is a $J-$set near zero if and only if for each $F \in P_f(\tau_0)$ and $\delta > 0$, there exist $a \in S \cap (0,\delta)$ and $H \in P_f(\mathbb{N})$ such that for each $f \in F, a + \sum_{t\in H} f(t) \in A \cap (0,\delta)$. for each $f \in F$, i.e. for each $\delta > 0, A \cap (0,\delta)$ is a $J-$set.
(3) $J(S) = \{p \in \beta S : \forall A \in p, A \,\text{is a}\, J\text{-set}\}$. Here $\mathcal{T} = \{f \mid f : \mathbb{N} \to S\}$.

Lemma 8. *Let S be a dense subsemigroup of $((0,+\infty),+)$ and let $A \subseteq S$. Then A is a $J-$set near zero if and only if whenever $F \in P_f(\tau_0)$ and $\delta > 0$, there exist $a \in S \cap (0,\delta)$ and whenever $m \in \mathbb{N}$,then $H \in P_f(\mathbb{N})$,with $\min H > m$,such that for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.*

Proof. [?] Lemma 14.8.2 . □

Theorem 9. *Let S be a dense subsemigroup of $((0,+\infty),+)$ and let $A \subseteq S$. If A is a piecewise syndetic set near zero, then A is a J-set near zero.*

Proof. [?]Theorem 3.4.

For more details on Central Sets theorem till date, see [?]. □

Now we can state the Central Sets theorem along Dev Phulara's way which is the following,

Theorem 10. *Let, $(S,+)$ be a commutative subsemigroup. Let r be an idempotent in $J(S)$, If $(C_n)_{n=1}^{\infty}$ be a sequence of central subsets in r ,*

then there exists $\alpha : P_f(\tau) \to S$, $H : P_f(\tau) \to P_f(\mathbb{N})$ such that,

1. If $F, G \in P_f(\tau), F \subset G$, then $\max H(F) < \min H(G)$.

2. If $t \in \mathbb{N}$ and $G_1, G_2, ..., G_t \in P_f(\tau)$, $G_1 \subset G_2 \subset ..... \subset G_t$ and $f_i \in G_i, i = 1, 2, ..., t$. If $|G_1| = m$, then

$\sum_{i=1}^{t} \alpha(G_i) + \sum_{t\in H(G_i)} f_i(t) \in C_m$.

Proof. [?] Thorem 2.6. □

Corollary 11. *Let $(S,+)$ be a commutative semigroup, let r be an idempotent in $J(S)$, If $(C_n)_{n=1}^{\infty}$ be central subsets in r. For each $l \in \{1,...,k\}, \{y_{l,n}\} \in \mathcal{T}$, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in S and a sequence $\{H_n\}_{n=1}^{\infty}$ in $P_f(\mathbb{N})$ such that $maxH_n < minH_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $F \in P_f(\tau)$ with $min\, F = m$,*

$\sum_{n\in F} \alpha_n + \sum_{t\in H_n} {}_i y_{i,t} \in C_m$.

Proof. [?] Theorem 2.8 □

Theorem 12. *Let $(S,+)$ be a commutative semigroup and let, A be $u \times v$ matrix satisfies the first entries ondition. Assume that for each first entry c of A, cS is a central*set . Let, $(C_n)_{n\in\mathbb{N}}$ be central subsets of S . Then for $i = 1, 2, ..., v$, there exist sequences $\{x_{i,n}\} \in \mathcal{T}$ in S . For every $F \in P_f(\mathbb{N})$, and if $m = min\, F$,*

$$\text{then } A\overrightarrow{x_F} \in (C_m)^u, \text{ where } \overrightarrow{x}_F \in \begin{pmatrix} \sum\limits_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum\limits_{n\in F} x_{v,n} \end{pmatrix} \in (S-\{0\})^v.$$

Proof. [?],Theorem 2.11. □

2. Generalized Central Sets Theorem Near 0 Along Phulara's Way

Theorem 13. *Let, S be a dense subsemigroup of $((0,\infty),+)$. If $(C_n)_{n=1}^{\infty}$ be central subsets near zero in S. Then for each $\delta \in (0,1)$,there exists $\alpha_\delta : P_f(\tau_0) \to S, H : P_f(\tau_0) \to P_f(\mathbb{N})$*

1. $\alpha_\delta < \delta$ for each $F \in P_f(\tau_0)$.

2. If $F, G \in P_f(\tau_0), F \subset G$, then $maxH_\delta(F) < minH_\delta(G)$.

3. If $m \in \mathbb{N}$and $G_1, G_2, ..., G_m \in P_f(\tau_0), G_1 \subset G_2 \subset ..... \subset G_m$and $f_i \in G_i, i = 1, 2, ..., m$. If $|G_1| = r$, then $\sum_{i=1}^{m}(\alpha_\delta(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in C_r$.

Proof. Pick a minimal idempotent p of 0^+ such that $A \in p$,

Let, $A^* = \{x + A : -x + A \in p\}$, so, $A^* \in p$ and $x \in A^* \Rightarrow -x + A^* \in p$

We define $\alpha_\delta(F) \in S, H_\delta(F) \in P_f(\mathbb{N})$ for $P_F(\tau_0)$ by induction on $|F|$, satisfying

1. $\alpha_\delta < \delta$ for each $F \in P_f(\tau_0)$.

2. If $F, G \in P_f(\tau_0), F \subset G$, then $maxH_\delta(F) < minH_\delta(G)$.

3. If $m \in \mathbb{N}$and $G_1, G_2, ..., G_m \in P_f(\tau_0), G_1 \subset G_2 \subset ..... \subset G_m$and $f_i \in G_i, i = 1, 2, ..., m$. If $|G_1| = r$, then

$\sum_{i=1}^{m}(\alpha_\delta(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in C_r^*$.

If $F = \{f\}$, C_1^* is piecewise syndetic near zero, Then C_1^* is J-set near zero.

Then $\forall\delta > 0$,there exists $a \in S \cap (0,\delta)$ and $L \in P_f(\mathbb{N})$ such that

$$a + \sum_{t\in L} f(t) \in C_1^*.$$

Let, $\alpha_\delta(\{f\}) = a, H_\delta(\{f\}) = L$. Then all three hypothesis are satisfied. Let, $|F| = n > 1$,now, $\alpha_\delta(G), H_\delta(G)$ have been defined for all proper subsets G of F.

Let, $K_\delta = \cup\{H_\delta(G) : G \subset F\} \in P_f(\tau_0)$. Let, $maxK_\delta = d$. Let,

$$M_{\delta_r} = \{\sum_{i=1}^{n}(\alpha_\delta(G_i) + \sum_{t\in H_\delta(G_i)} f_i(t)) : n \in \mathbb{N}, \emptyset \neq G_1 \subset G_2 \subset .... \subset G_n \subset F, f_i \in G_i, \forall i = 1, 2, .., n. |G_1| = r\}$$

M_{δ_r} is finite and by hypothesis (3), $M_{\delta_r} \subseteq C_r^*$. Let

$$B = C_n^* \cap (\cap_{x \in M_{\delta_r}} (-x + C_r^*))$$

then, $B \in p$,and so B is a J-set near zero.

Then $\forall \delta > 0$,there exists $a \in S \cap (0, \delta)$ and $L \in P_f(\mathbb{N})$ with $min\, L > d$ such that

$$a + \sum_{t \in L} f(t) \in B, \forall f \in F.$$

Let, $\alpha_\delta(\{f\}) = a, H_\delta(\{f\}) = L$.

Hypothesis (1) obvious, Since, $min\, L > d$, hypothesis (2) is satisfied. Now remaining to check hypothesis (3),

Pick $\delta > 0, n \in \mathbb{N}$. Let, $\emptyset \neq G_1 \subset G_2 \subset ..... \subset G_n = F$. If, $n = 1$, then $G_1 = F$,i.e, $r = n$.

$$a + \sum_{t \in L} f(t) = \alpha_\delta(F) + \sum_{t \in H_\delta(F)} f(t) \in B \subseteq C_n^*$$

Now, $n > 1$, and let $y = \sum_{i=1}^{n-1} (\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} f_i(t))$, $f_i \in G_i$. Therefore, $y \in M_{\delta_r}$ and

$$a + \sum_{t \in L} f(t) \in B \subseteq -y + C_r^*$$

which implies,

$$\sum_{i=1}^{n} (\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} f_i(t)) \in C_r^*.$$

□

Theorem 14. *Let, S be a dense subsemigroup of $((0, \infty), +)$. Let $(C_n)_{n=1}^\infty$ be central subsets near zero in S. For each $l \in \{1, ..., k\}, \{y_{l,n}\} \in \mathcal{T}_0$, there exists a sequence $\{\alpha_n\}_{n=1}^\infty$ in S such that $\alpha_n \to 0$ and a sequence $\{H_n\}_{n=1}^\infty$ in $P_f(\mathbb{N})$ such that $maxH_n < minH_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $F \in P_f(\mathcal{T}_0)$ with $min\, F = m$,*

$$\sum_{n \in F} \alpha_n + \sum_{t \in H_n} y_{l,t} \in C_m$$

Proof. We may assume $C_n \subseteq C_{n+1}, \forall n$

Let, $\gamma_u \in P_f(\mathcal{T}_0) - \{\{y_{1,n}\}, ..., \{y_{l,n}\}\}$ such that $\gamma_u \neq \gamma_v$,if $u \neq v$.

$G_u = \{\{y_{1,n}\}, ..., \{y_{l,n}\}\} \cup \{\gamma_1, \gamma_2, ...\gamma_u\} \in P_f(\mathcal{T}_0)$,

$\alpha_\delta(\{G_u\}) = a_u$, $H_\delta(\{G_u\}) = H_u$

Let, $F \in P_f(\mathbb{N})$ be enumerated in order as $F = \{n_1, ..., n_s\}$.

Since,$\alpha_n \to 0$ therefore, $\alpha_\delta(G) < \delta$ for each $G \in P_f(\tau_0), \forall \delta > 0$ and $G_n \subset G_{n+1}, \forall n$,therefore, $maxH_n < minH_{n+1}$ for each $n \in \mathbb{N}$.

$\emptyset \neq G_{n_1} \subset G_{n_2} \subset ..... \subset G_{n_s}, |G_{n_1}| = n_1 + l$

therefore, $\sum_{i=1}^{s} (\alpha_\delta(G_i) + \sum_{t \in H_\delta(G_i)} y_{l,t} \in C_{n_1+1} \subseteq C_{n_1}$.

Hence, $\sum_{n \in F} \alpha_n + \sum_{t \in H_n} y_{l,t} \in C_{n_1}$. □

Theorem 15. *Let, S be a dense subsemigroup of $((0, \infty), +)$ and let, A be $u \times v$ matrix satisfies the first entries ondition. Assume that for each first entry c of A, cS is a central* set near zero. Let, $(C_n)_{n \in \mathbb{N}}$ be central subsets of S near zero. Then*

for $i = 1, 2, ..., v$, *there exist sequences* $\{x_{i,n}\} \in \mathcal{T}_0$ *in* S . *For every* $F \in P_f(\mathbb{N})$, *and if* $m = min\, F$,*then* $A\overrightarrow{x_F} \in (C_m)^u$, *where* $\overrightarrow{x}_F \in \begin{pmatrix} \sum_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum_{n\in F} x_{v,n} \end{pmatrix} \in (S - \{0\})^v$.

Proof. We proceed by induction on v. Assume first that $v = 1$. We can assume A has no repeated rows. In the case we have $A = (c)$ for some cS is a $central^*$set near zero and $(C_n \cap cS)_{n\in\mathbb{N}}$ is central subsets of S near zero. with $(C_n \cap cS) \subseteq (C_{n+1} \cap cS)$. Pick a sequence $(l_n)_{n\in\mathbb{N}}$such that $l_n \to 0$,for every $F \in P_f(\mathbb{N}), m = minF$, then $\sum_{n\in F} l_n \in C_m \cap cS$. For each $n \in \mathbb{N}$,pick $x_{1,n} \in S$ such that $l_n = cx_{1,n}$. The sequence $(x_{1,n})_{n\in\mathbb{N}}$ is as required.

Now assume $v \in \mathbb{N}$ and the theorem is true for v. Let, A be a $u \times (v+1)$ matrix with entries from ω which satisfies the first entries condition, and assume that for every first entry c of A, cS is a central* set near zero. By rearranging rows of A and adding additional rows of A if needed, we may assume that we have some $t \in \{1, 2, .., u-1\}$ and $d \in \mathbb{N}$ such that

$$a_{i,1} = \begin{array}{l} 0, \text{ if } i \in \{1, 2, ..., t\} \\ d, \text{ if } i \in \{t+1, ..., u\} \end{array}$$

Let, B be the $t \times v$ matrix with entries $b_{i,j} = a_{i,j+1}$. Pick sequences $\{z_{1,n}\}_{n\in\mathbb{N}}, ..., \{z_{v,n}\}_{n\in\mathbb{N}}$ in S as guaranteed by induction hypothesis for the matrix B. For each $i \in \{t+1, ..., u\}$ and each $n \in \mathbb{N}$, let

$$y_{1,n} = \sum_{j=2}^{v+1} a_{i,j} z_{j-1,n}$$

and let, $y_{t,n} = 0, \forall n \in \mathbb{N}$. and $y_{i,n} \in S$.

Now, $(C_n \cap dS)_{n\in\mathbb{N}}$ is central subsets of S near zero. Pick a sequence $(l_n)_{n\in\mathbb{N}}$such that $l_n \to 0$ in S, a sequence $\{H_n\}_{n=1}^{\infty}$ in $P_f(\mathbb{N})$ such that $maxH_n < minH_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $i \in \{t, t+1, ..., u\}$ and for all $F \in P_f(\mathcal{T}_0)$ with $min\, F = k$, then

$$\sum_{n\in F}(l_n + \sum_{t\in H_n} y_{l,t}) \in C_k \cap dS$$

Note , in particular that, with $F = \{n\}$,

$$l_n = (l_n + \sum_{t\in H_n} y_{l,t}) \in C_n \cap dS$$

So pick $x_{1,n} \in S$ such that $l_n = dx_{1,n}$. For, $j \in \{2, 3, ..., v+1\}, n \in \mathbb{N}$, let $x_{j,n} = \sum_{s\in H_n} z_{j-1,n}$. We claim that the sequence $(x_{j,n})_{n\in\mathbb{N}}$ is as required. To see this , let $F \in P_f(\mathcal{T}_0)$ with $min\, F = k$. We need to show that for $j \in \{1, 2, ..., v+1\}$, $\sum_{n\in F} x_{j,n} \neq 0$, and for each $i \in \{1, 2, ..., u\}$,

$$\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n} = \sum_{j=2}^{v+1} a_{i,j} \sum_{n\in F} \sum_{s\in H_n} z_{j-1,s} = \sum_{j=1}^{v} b_{i,j} \sum_{s\in H_n} z_{j,s}$$

where $H = \cup_{n\in F} H_n$. Let, $k^{'} = min\, G$. Then $k \leq k^{'}$, so by the induction hypothesis,

$\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n}$
$= a_{i,1} \sum_{n\in F} x_{i,n} + \sum_{j=2}^{v+1} a_{i,j} \sum_{n\in F} \sum_{s\in H_n} z_{j-1,s}$
$= d \sum_{n\in F} x_{1,n} + \sum_{j=2}^{v+1} a_{i,j} \sum_{n\in F} \sum_{s\in H_n} z_{j-1,s}$
$= \sum_{n\in F} dx_{1,n} + \sum_{n\in F} \sum_{s\in H_n} \sum_{j=2}^{v+1} a_{i,j} z_{j-1,s}$
$= \sum_{n\in F}(l_n + \sum_{s\in H_n} y_{i,s}) \in C_k.$ □

3. Some more general versions along a sequence of Central subsets

Now we want to state another version of Central sets theorem which was studied by De et al.[?]

Theorem 16. *Let, $(S,+)$ be a discrete commutative semigroup, let $c \in \mathbb{N}$, let $\mathcal{A} \subseteq P(S)$, and $T = \cap_{A\in\mathcal{A}} cl(A)$. Let $(D,\leq)$ be a directed set, and let $\mathcal{T} \subseteq D_S$, the set of functions from D to S.*
Assume that
(1) $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$,
(2) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq \mathcal{A})$,and
(4) $(\forall A \in \mathcal{A})(\forall d \in D)(\forall F \in P_f(\mathcal{T}))(\exists d' \in D)(d < d', \forall f \in F)(f(d') \in A) b_i = \vec{r_i}(\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m})) \in Q$
Assume that $K(T) \cap \bigcap_{A\in\mathcal{A}} cl(cA) \neq \emptyset$, and $p \in K(T) \cap \bigcap_{A\in\mathcal{A}} cl(cA)$, and $C_n \in p, \forall n \in \mathbb{N}$ then $\forall F \in P_f(\tau)$,$d \in D$,$A \in \mathcal{A}$, $\exists a \in A, m \in \mathbb{N}, d_1, .., d_m$ in D such that $d_1 < ... < d_m, ca \in C$, and for all $f \in F$, $ca + \sum_{j=1}^{m} f(d_j) \in C$.

Proof. [?] Lemma 3.2. □

Now we want to rephrase this version towards the way Phulara Generalized the Central Sets theorem. Our statement will be the following,

Theorem 17. *Let, $(S,+)$ be a discrete commutative semigroup. Let, $\mathcal{A} \subseteq P(S)$, and let $T = \cap_{A\in\mathcal{A}} cl(A)$, $(D,\leq)$ be a directed set, and let $\mathcal{T} \subseteq D_S$, the set of functions from D to S.*
Assume that
(1) $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$,
(2) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq \mathcal{A})$,and
(4) $(\forall A \in \mathcal{A})(\forall d \in D)(\forall F \in P_f(\mathcal{T}))(\exists d' \in D)(d < d', \forall f \in F)(f(d') \in A) b_i = \vec{r_i}(\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m})) \in Q$
Let, $(C_n)_{n=1}^{\infty} \in p$, p is an idempotent with $p \in cl(C_n) \cap K(T), \forall n$.
Let, $\Phi : P_f(\mathcal{T}) \to \mathcal{A}$, Then $\exists \alpha : P_f(\mathcal{T}) \to C$ & $H : P_f(\mathcal{T}) \to P_f^{lin}(D)$ such that
(a) $\forall F \in P_f(\mathcal{T})(\alpha(F) \in \phi(F))$
(b) $\forall F \in P_f(\mathcal{T})(\forall f \in F)(\alpha(F) + \sum_{t\in H(F)} f(t) \in C \cap \phi(F)$
(c) If $F, G \in P_f(\mathcal{T}), G \subset F$, then $max\, H(G) < min\, H(F)$
(d) If $m \in \mathbb{N}$,$G_1, ..., G_m \in P_f(\mathcal{T}), G_1 \subset .... \subset G_m, f_i \in G_i$, if $r = |G_1|$,

$$\sum_{i=1}^{m} (\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in C_r$$

Proof. Let $C_n^* = \{s \in C_n : -s + C_n \in p\}$. We may assume $C_{n+1} \subseteq C_n, \forall n \in \mathbb{N}$.
Therefore, $x \in C_n^* \Rightarrow -x + C_n^* \in p$.

We define $\alpha(F)$ & $H(F)$ by induction on $|F|$ such that
(i) $\alpha(F) \in \phi(F) \cap C_1^*$
(ii) $(\forall f \in F)(\alpha(F) + \sum_{t\in H(F)} f(t) \in C_1^* \cap \phi(F)$
(iii) If $F, G \in P_f(\mathcal{T}), \emptyset \neq G \subset F$, then $max\, H(G) < min\, H(F)$
(iv) If $m \in \mathbb{N}$,$G_1, ..., G_m \in P_f(\mathcal{T})$, $G_1 \subset .... \subset G_m$, $f_i \in G_i$, if $r = |G_1|$,

$$\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in C_r^*$$

If $F = \{f\}$,By Theorem 16, $\alpha(F) \in \phi(F) \cap C^*, m \in \mathbb{N}$ & $d_1, ..., d_m \in D$, $d_1 < ... < d_m$ & $\alpha(F) + \sum_{i=1}^{m} f(d_i)) \in C_1^* \cap \phi(F)$.

Let, $H(F) = \{d_1, ..., d_m\}$ and all hypothesis hold.

Let, $|F| = n > 1$,now, $\alpha(G), H(G)$ have been defined for all proper subsets G of F.

Let, $L = \cup\{H(G) : G \subset F\}$, pick $d \in D$ such that $F, G \in P_f(\mathcal{T}), \emptyset \neq G \subset F$,
Then $max\, H(G) = d$. Let,

$$M_r = \{\sum_{i=1}^{r}(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) : r \in \mathbb{N}, \emptyset \neq G_1 \subset G_2 \subset .... \subset G_n \subset F, f_i \in G_i, \forall i = 1, 2, .., n. |G_1| = r\}.$$

by hypothesis (iv), $M_r \subseteq C_r^*$.
Let $B = C_n^* \cap \phi(F) \cap (\cap_{x\in M_r}(-x + C_r^*))$
then $B \in p$, by Theorem 16, $\alpha(F) \in \phi(F) \cap C_1^*$,$n \in \mathbb{N}$ & $e_1, ..., e_m \in D$ such that $e_1 < ... < e_n$ & for each $f \in F$, $\alpha(F) + \sum_{i=1}^{m} f(e_i)) \in B$.

Let, $H(F) = \{e_1, ..., e_n\}$, Then hypothesis (i) and (ii) satisfied.

$F, G \in P_f(\mathcal{T}), \emptyset \neq G \subset F$,then $max\, H(G) \leq d < e_1 \leq min\, H(F)$, therefore (iii) holds.

To verify (iv) , let, $m \in \mathbb{N}$,$G_1, ..., G_m \in P_f(\mathcal{T})$, $\emptyset \neq G_1 \subset .... \subset G_m$, $f_i \in G_i$, $i \in \{1, ..., m\}$ assume first $m = 1, r = |G_1| = |F| = n$, (i.e.$G_1 = F$)
$\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) = \alpha(G_1) + \sum_{t\in H(G_1)} f_1(t) \in B \subseteq C_n^* = C_r^*$
Now assume $m > 1$,$x = \sum_{i=1}^{m-1}(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in M_r$
therefore $\alpha(F) + \sum_{t\in H(F)} f_m(t)) \in B \subseteq -x + C_r^*$
Which means $\alpha(F) + \sum_{t\in H(F)} f_m(t) + x \in C_r^*$, that is
$\sum_{i=1}^{m}(\alpha(G_i) + \sum_{t\in H(G_i)} f_i(t)) \in C_r^*$. □

Theorem 18. *Let, $(S, +)$ be a discrete commutative semigroup. Let, $\mathcal{A} \subseteq P(S)$ let,$u, v \in \mathbb{N}$, M be a $u \times v$ first entries matrix with entries from ω & assume that*
(1) $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$,
(2) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq \mathcal{A})$,and
(4) $(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B + B \subseteq \mathcal{A})$
$T = \cap_{A\in\mathcal{A}} cl(A)$, Then T is a subsemigroup of βS_d. If p is an idempotent in $K(T) \cap \bigcap_{A\in\mathcal{A}} \bigcap_{c\in P(M)} cl(cA)$and $C_n \in p, \forall n \in \mathbb{N}$, Then $\forall A \in \mathcal{A}$,

$\exists$ *sequence* $(x_{1,n}), ..., (x_{v,n})$ *in* A. *such that* $\vec{x}(\vec{G}) \in \begin{pmatrix} \sum\limits_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum\limits_{n\in F} x_{v,n} \end{pmatrix}, F \in P_f(\mathbb{N})$

,with $min\, F = r$,

Then $M_2\vec{x}(\vec{G}) \in (C_r)^u$.

Proof. Assume that p is an idempotent $K(T) \cap \bigcap_{A\in\mathcal{A}} \bigcap_{c\in P(M)} cl(cA)$ and $C_n \in p, \forall n \in \mathbb{N}$.

$C_n^*(p) = \{s \in C_n : -s + C_n \in p\}$

We may assume that M has no repeated rows. We proceed by induction on v.

If $v = 1$, then $M = (c)$, Given $A \in \mathcal{A}$, $cA \cap C_n \in p, \forall n$.

A sequence $(l_n)_{n=1}^{\infty}$ such that for every $F \in P_f(\mathbb{N}), k = min\, F$, Then $\sum_{n\in F} l_n \in cA \cap C_k$

Pick $x_{1,n}$ such that $l_n = cx_{1,n}$, where $x_{1,n} \in A, \forall n$.

Therefore $Mx \in C_k$, where $M = (c)$, $x = \sum_{n\in F} x_{1,n}$.

Now, Let $v \in \mathbb{N}$ & assume the result valid for v. Let, M be a $u \times (v+1)$ matrix. By rearranging rows & adding rows if need be, we have

$$M = \begin{pmatrix} C & \bar{0} \\ \bar{C} & M_1 \\ \bar{0} & M_2 \end{pmatrix}$$

where M_1 & M_2 are matrix entries from w.

M_1 is $u_1 \times v$ matrix & no row equal to $\vec{0}$ & M_2 is $u_2 \times v$ first entries matrix . Also $P(M) = P(M_1) \cap \{c\}$.

For $i \in \{1, 2, ..., u-1\}$, let $\vec{r_i}$ be the i th row of $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$.

Let, D be the set of finite sequence in A, ordering D by $\vec{G} < \vec{G}'$ iff $\vec{G}$ is a proper subsemigroup of $\vec{G}'$.

Given $\vec{G} \in D$, Let, $\vec{x}(\vec{G}) \in S^v$ for each $\vec{G} \in D$. So that

(a) $\vec{r_i}.\vec{x}(\vec{G}) \in \cap_{j=1}^{l(\vec{G})} G_j, \forall i \in \{1, 2, ..., u_1\}$

(b) if $m \in \mathbb{N}, \vec{G_1}, ..., \vec{G_m} \in D$ & $\vec{G_1} < .... < \vec{G_m}$ with $l(\vec{G_1}) = k$, then

(i) $M_2(\vec{x}(\vec{G_1}) + ... + \vec{x}(\vec{G_m})) \in (C_k^*)^{u_2}$

(ii) $\vec{x}(\vec{G_1}) + ... + \vec{x}(\vec{G_m}) \in (\bigcap_{i=1}^{l(\vec{G_1})} G_{1,i})$

We proceed by induction on $l(\vec{G})$. Assume first that $\vec{G} = \langle A \rangle$, Pick by (4) , some $B \in \mathcal{A}$ such that $\forall \vec{y} \in B^v, M_1.\vec{y} \in A^u$,

since $A \cap B \in \mathcal{A}$, if $\vec{x}(G) \in (A \cap B)^v$ & $\exists x_{1,n}, ..., x_{v,n}$ such that

$$\vec{x}(\vec{G}) \in \begin{pmatrix} \sum\limits_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum\limits_{n\in F} x_{v,n} \end{pmatrix}, \; min\, F = r,$$

such that $M_2\vec{x}(\vec{G}) \in (C_r^*)^{u_2}$ (by induction)

Now , assume that $l(\vec{G}) > 1$ & we have choosen $\vec{x}(\vec{H})$ for every proper subsequence $\vec{H}$or $\vec{G}$.

Let, $A = \cap_{j=1}^{l(\vec{G})} G_j$, Let

$R = \{(\vec{H_1}, .., \vec{H_k}) : k \in \mathbb{N}, \vec{H_1}, .., \vec{H_k} \in D, l(H_1) = r, \text{and} \vec{H}_1 < ... < \vec{H_k} < \vec{G}\}$

Given $\mathcal{H} = (\vec{H_1}, ..., \vec{H_k}) \in R$, We have $\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m}) \in (\bigcap_{i=1}^{l(\vec{H_1})} H_{1,i})^v$

& $\cap_{i=1}^{l(\vec{G_1})} H_{1,i} \in \mathcal{A}$. Applying assumtion (1) and (4) pick $D_{\mathcal{H}} \in \mathcal{A}$ such that

$\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m}) + (D_{\mathcal{H}})^v \subseteq (\bigcap_{i=1}^{l(\vec{H_1})} H_{1,i})^v$

Pick by lemma 4, some $B \in \mathcal{A}$ such that $\forall \vec{y} \in B^v, M_1\vec{y} \in A^{u_1}$, let $E = A \cap B \bigcap_{\mathcal{H}\in R} D_{\mathcal{H}}$

Let, $Q_r = \{\vec{r_i}.(\vec{x}(\vec{H_1}) + ...\vec{x}(\vec{H_k})) : (\vec{H_1}, ..., \vec{H_k}) \in R \text{ and } i \in \{u_1 + 1, ..., u - 1\}\}$

Then $Q_r \subseteq C_p^*$. Pick $\vec{x}(\vec{G}) \in E^v$, $M_2\vec{x}(\vec{G}) \in (C_n^* \cap \bigcap_{p=1}^{n-1} \bigcap_{b\in Q_r}(-b + C_r^*) = A$

Since, $\vec{x}(\vec{G}) \in B^v$, hypothesis (a) hold. where $\vec{x}(\vec{G}) \in \begin{pmatrix} \sum\limits_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum\limits_{n\in F} x_{v,n} \end{pmatrix}$, $A \in p$.

To verify hypothesis (b), let $m \in \mathbb{N}$, let, $\vec{H_1}, ..., \vec{H_m} \in D$ and $\vec{H_1} < .... < \vec{H_m} = \vec{G}$

If $m = 1$, we have $\vec{x}(\vec{G}) \in A^v$, $M_2\vec{x}(\vec{G}) \in (C_r^*)^{u_2}$,

Where $\vec{x}(\vec{G}) = \begin{pmatrix} \sum\limits_{n\in F} x_{1,n} \\ . \\ . \\ . \\ \sum\limits_{n\in F} x_{v,n} \end{pmatrix}$,$min\, F = r$ as required.

So, assume $m > 1$,& $\mathcal{H} = (\vec{H_1}, ..., \vec{H_k}) \in R$, then $\vec{x}(\vec{G}) \in (D_{\mathcal{H}})^v$ so

$\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m}) + (D_{\mathcal{H}})^v \subseteq (\bigcap_{i=1}^{l(\vec{H_1})} H_{1,i})^v$

And for $i \in \{u_1 + 1, ..., u - 1\}$,

$b_i = \vec{r_i}(\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_m})) \in Q$

So, $\vec{r_i}.\vec{x}(\vec{G}) \in -b + C^*$ and thus

$\vec{r_i}(\vec{x}(\vec{H_1}) + ... + \vec{x}(\vec{H_k}) + \vec{x}(\vec{G})) \in C^*$

Now for $i \in \{1, ..., u_1\}$,define $f_i(\vec{G}) = \vec{r_i}.\vec{x}(\vec{G})$.

Let, $\mathcal{T} = \{f_1, .., f_{u_1}\}$.We claim that hypotheses of Theorem 16 are satisfied . Hypothesis (1),(2) and (3) are satisfied directly.

To verify hypothesis (4) ,let $A \in \mathcal{A}$, $\vec{G} = \langle G_1, ..., G_k\rangle \in D$ and $F \in P_f(\mathcal{T})$ be given . Let, $\vec{G}' = \langle G_1, ..., G_k, A\rangle$.

Then $\vec{G} < \vec{G}'$ and if $f_i \in F$, then $f_i(\vec{G}') = \vec{r_i}.\vec{x}(\vec{G}') \in A$. (by condition (a)).

To conclude the proof , let $A \in \mathcal{A}$, and let $\vec{G} = \langle A\rangle$, Pick by lemma (4),

$a \in A, m \in \mathbb{N}$,and $\vec{G_1}, ..., \vec{G_m} \in D$ and $\vec{G_1} < .... < \vec{G_m}, l(\vec{G_1}) = r$, $ca \in C_r$ and for all $f \in F$,

$$ca + \sum_{j=1}^{m} f(\vec{G_j}) \in C_r$$

Let, $\vec{y} = \vec{x}(\vec{G_1}) + ... + \vec{x}(\vec{G_m})$.

Then $\vec{y} \in (A)^v$ where$\vec{y} = \begin{pmatrix} \sum_{n\in F} y_{1,n} \\ . \\ . \\ . \\ \sum_{n\in F} y_{v,n} \end{pmatrix} \in A^v$,$min\, F = r$ and $M_2\vec{y} \in (C_r^*)^{u_2}$.

If $i \in \{1, ..., u_1\}$,then $ca + \vec{r_i}.\vec{y} = ca + \sum_{j=1}^{m} f_i(\vec{G}) \in C_r$, where $(a_{1,n})_{n\in\mathbb{N}} \in A$, and $a = \sum_{n\in F} a_{1,n}$.

So, $\begin{pmatrix} a \\ \vec{y} \end{pmatrix} \in A^{v+1}$ and $M \begin{pmatrix} a \\ \vec{y} \end{pmatrix} \in C_r^u$. □

Lemma 19. *Let A be a piecewise $\mathcal{F}$-syndetic set and $F \in \mathcal{P}_f^{\mathcal{F}}\left({}^{\mathbb{N}}S\right)$ be $\mathcal{F}$- good. Then there exist $a_1, a_2, \ldots, a_{\ell+1} \in S$ and $\{h_1, h_2, \ldots, h_\ell\} \subset \mathbb{N}$ such that,*

$$a_1 f(h_1)\, a_2 f(h_2) \ldots a_\ell f(h_\ell)\, a_{\ell+1} \in A$$

for all $f \in F$.

Proof. [?] Lemma 2.2. □

Lemma 20. *Let $(S, \cdot)$ be a semigroup and $\mathcal{F}$ be a filter such that $\bar{\mathcal{F}}$is a semigroup. Let A be a $\mathcal{F}$-J set and $F \in \mathcal{P}_f^{\mathcal{F}}\left({}^{\mathbb{N}}S\right)$. Then for all $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $a \in S^{l+1}$, $t \in \mathcal{J}_l$ and $x(m, a, t, f) = a_1 f(t(1))\, a_2 f(t(2)) \cdots a_m f(t(m))\, a_{m+1} \in A$ for all $f \in F$ where $t(1) > m$.*

Proof. [?]Lemma 2.3. □

Now we are ready to generalize Central sets theorem along a filter along Phulara's way, the motivation comes from [?]. Here is the statement,

Theorem 21. *Let $(S, .)$ be a semigroup and $\mathcal{F}$ be a filter on S such that $\bar{\mathcal{F}}$ is a semigroup. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$ central set on S. Then there exist functions $m : P_f^{\mathcal{F}}(\tau) \to \mathbb{N}$,and $\alpha \in \times_{F\in\mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S)} S^{m(F)+1}$ and $\tau \in \times_{F\in\mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S)} \mathcal{J}_{m(F)}$ such that*

1. If $F, G \in \mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S), G \subset F$, then $\tau(G)(m(G)) < \tau(F)(1)$. and

2. If $n \in \mathbb{N}$and $G_1, G_2, ..., G_t \in \mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S), G_1 \subset G_2 \subset ..... \subset G_n$and $f_i \in G_i, i = 1, 2, ..., n$. If $|G_1| = m$, then

$$\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), \tau(G_i), f_i) \in C_m$$

Proof. Let $\mathcal{F}$ be a filter on S such that $\bar{\mathcal{F}}$ is a semigroup. Assume $(C_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$ central set on S. Then there exists a minimal idempotent of $\bar{\mathcal{F}}$ say,$p \in K(\bar{\mathcal{F}})$ such that $A \in p$. as $p.p = p$,so$A^* = \{x + A : -x + A \in p\}$, so, $A^* \in p$ and $x \in A^* \Rightarrow -x + A^* \in p$. Now define $m(F) \in \mathbb{N}$, $\alpha(F) \in S^{m(F)+1}$and $\tau(F) \in \mathcal{J}_{m(F)}$, by induction on $|F|$, satisfying the following inductive hypothesis:

1. If $F, G \in \mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S), G \subset F$, then $\tau(G)(m(G)) < \tau(F)(1)$ and
2. If $n \in \mathbb{N}$and $G_1, G_2, ..., G_n \in \mathcal{P}_f^{\mathcal{F}}(\mathbb{N}_S), G_1 \subset G_2 \subset ..... \subset G_n$and $f_i \in G_i, i = 1, 2, ..., t$. If $|G_1| = m$, then

$$\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), \tau(G_i), f_i) \in C_m^*$$

□

Assume first that$F = \{f\}$. As C_1^* is piecewise $\mathcal{F}$- syndetic , pick $m \in \mathbb{N}$,$a \in S^{m+1}$, $t \in \mathcal{J}_m$ such that

$$x(m, \alpha, t, f) \in C_1^*$$

So the hypothesis is satisfied vacuously. Now assume$|F| > 1$, and $m(F)$, $\alpha(F)$and $\tau(F)$ have been defined for all proper subsets G of F. Let, $K = \{\tau(G) : \emptyset \neq G \subset F\}$,and $m = max_{\emptyset\neq G\subset F}\tau(G)(m(G))$. Let ,

$$M_r = \{\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), \tau(G_i), f_i) : n \in \mathbb{N}, \emptyset \neq G_1 \subset G_2 \subset .... \subset G_n \subset F, f_i \in G_i, \forall i = 1, 2, .., n.|G_1| = r\}$$

M_r is finite and by hypothesis (2), $M_r \subseteq C_r^*$ by induction. Let,

$$B = C_n^* \cap (\cap_{x\in M_r}(x^{-1}C_m^*))$$

Let $B \in p$, by the lemma 19 and 20 there exist $N \in \mathbb{N}$,$a \in S^{N+1}$, $t \in \mathcal{J}_N$ and $\tau(1) > m$ such that $x(m, \alpha, t, f) \in B$ for all $f \in F$.

We write $m(F) = N$, $\alpha(F) = a$and $\tau(F) = t$.Then clearly $\tau(F)(1) = t(1) > \tau(G)(m(G))$ for all $\emptyset \neq G \subset F$. Denote $G_{n+1} = F$, so the first hypothesis of the induction is satisfied. Now remaining to satisfy hypothesis (2). Pick $\delta > 0, n \in \mathbb{N}$.

Let, $\emptyset \neq G_1 \subset G_2 \subset ..... \subset G_n = F$. If, $n = 1$, then $G_1 = F$,i.e, $r = n$

$$x(m, \alpha, t, f) \in B \subset C_n^*$$

now, $n > 1$, and let $y = \prod_{i=1}^{n-1} x(m(G_i), \alpha(G_i), \tau(G_i), f_i), f_i \in G_i$. therefore, $y \in M_r$, $x(m, \alpha, t, f) \in B \subseteq y^{-1}C_m^*$. then $x(m, \alpha, t, f)y \in C_m^*$, therefore

$$\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), \tau(G_i), f_i) \in C_m^*$$

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